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any real number (including 0). A complex number is the sum of a real number and of a pure imaginary one. Finally, an imaginary number is a complex number which is not real. Linguistically, we don't like to have imaginaries come later than pure imaginaries. Very well, say "neomonic," if you wish. The important thing is that there must be a name to cover all numbers of the complex plane, a second one for those on the vertical axis, and a third for all numbers not on the horizontal axis.

Now that I am offering suggestions to writers of text-books and dictionaries, I will venture on another remark. $\sqrt{x}$ is always called a single-valued function when x is positive or zero, it is often regarded as such when x is negative, but it is undoubtedly double-valued when x is—if I may use the definition in the last paragraph—imaginary. $\sqrt[3]{x}$ is single-valued if and only if x is real, $\sqrt[4]{x}$ only if x is positive. It is perhaps inevitable, it is surely bewildering, that the same symbol should indicate, now a single-valued, now a multiple-valued function. There is a need for a symbol which shall always indicate that we may take our choice among all the nth roots of x. Should we not agree that $x^{1/n}$ shall be that symbol? Should not future books say that $4^{1/2} = \pm \sqrt{4} = \pm 2$?

II. The Formula $\frac{1}{2}a\,(a + 1)$ for the Area of an Equilateral Triangle.

A Reply to Professor Miller by Florian Cajori, University of California.

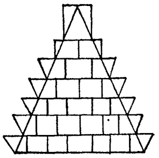
In this Monthly (*1921,* 257) Professor G. A. Miller writes on Gerbert's explanation of the question why $\frac{1}{2}a(a + 1)$ gives too large a value for the area of an equilateral triangle; Professor Miller claims that Cantor's figure is "inaccurate" and then states:

"What is more important is the fact that the corresponding figure found in various histories is still more misleading since it represents according to the explanations in the text an isosceles triangle whose base is equal to the altitude, while the text itself relates to an equilateral triangle. This fact can be verified by consulting either edition of Cajori's *History of Elementary Mathematics,* 1896 or 1917, p. 132."

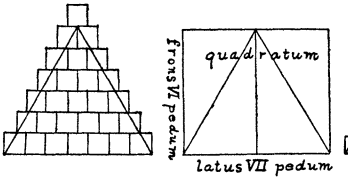
Professor Miller is in error; the figure in the 1917 edition does not represent "according to the explanations in the text an isosceles triangle whose base is equal to the altitude"; my figure, as well as my explanation, fit (as they should) the case of an equilateral triangle.

One is astonished at Professor Miller's declaration that Bubnov in his edition of Gerbert's *Opera mathematica* gives "a correct figure," and that those of Cantor, Günther and Cajori are all inaccurate or misleading. In the first place, Bubnov gives *two* figures. In the second place, Professor Miller misses completely the essential fact that we do not possess Gerbert's own drawing, and that the drawings in our histories are necessarily conjectural. Neither of Bubnov's two figures agrees with the figure given in Pez's edition of Gerbert's geometry and in the reprint of Pez by Migne in 1880. Pez's figure is due to some unskilful scribe of the Middle Ages whose copy of that geometry contains errors both in the text and in the figures.[1] In the scribe's drawing (which we here reproduce) the 28 small areas ("XXVIII pedes") are not all alike either in shape or size; hence that drawing is obviously wrong and does not correctly represent Gerbert's reasoning.

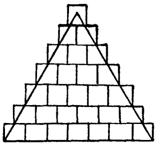
Gerbert's text is not sufficiently explicit to enable us to reconstruct the figure with certainty. Three somewhat different attempts have been made, one by Cantor, a second by Günther, a third by Bubnov. We reproduce all three.



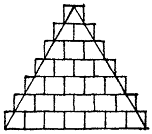
From MS. used by Pez



Bubnov's conjectures



Cantor's conjecture



Günther's conjecture

Bubnov's first figure represents Gerbert's rough approximation of 6 feet as the height of the equilateral triangle whose sides are 7 feet. Gerbert knew that 6 was only approximate. Nevertheless Bubnov exhibits the impossible figure in which the sides are 7 and the height is exactly 6.

Cantor's figure shows that the altitude really exceeds 6. For any equilateral triangle whose side is an integer exceeding 7, some of the squares will be wholly outside of the triangle and a very slight but obvious alteration of phraseology in Gerbert's letter would be the only change necessary to make the argument general.

In Günther's drawing, 28 rectangles are used instead of 28 squares. With Gerbert "quadratus" sometimes meant "rectangle." [2] These rectangles are together less in area than $\frac{1}{2}a(a+1)$, nevertheless they more than cover the triangle. The advantage of this construction is that it gives elegance to Gerbert's argument by being at once applicable to any equilateral triangle, be its sides 7 or 30 or a still larger integer. Professor Miller's declaration that Gerbert's argument is "trivial" because it cannot be extended to cases in which $a > 7$ is unwarranted by the facts.

Professor Miller misinterprets Hankel. When Hankel estimates Gerbert's letter as "the first mathematical paper of the Middle Ages which deserves this name," he means the Middle Ages *in Europe*. Hankel had considered Hindu and Arabic research in previous chapters. His statement is explicit; his "Mittelalter" refers, as he says, to the "abendländischen Nationen."

COMMENTS ON THE "REPLY" OF PROFESSOR CAJORI BY G. A. MILLER, University of Illinois.

The main question involved in the preceding "reply" seems to be whether square feet, or feet which are rectangles having unequal sides, were used in measuring the areas of the particular equilateral triangles under consideration, since the figures in question relate to the following statement by Gerbert: "Behold in this little diagram there are included 28 feet, not all of which are integral."[1] A little earlier in the same letter Gerbert said: "I believe it is known to you what feet are linear, what square, and what cubic, and that in measuring areas we take only square feet."

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[1] H. Weissenborn, *Gerbert*, Berlin, 1888, p. 42.

[2] See Gerbert's "Geometria" in J. P. Migne, editor, *Patrologiae cursus completus ecclesiae latinae*, vol. 139, Paris, 1880, p. 137. Gerbert, *Opera mathematica*, ed. by N. Bubnov, Berlin, 1899, p. 344.

One of the two rules given in this letter by Gerbert to find the area of an equilateral triangle was to multiply one half of the numerical measure of the base by the numerical measure of the altitude. There seems to be no question here as regards the kind of feet used, but, as I understand it, the author of the preceding "reply" aims to convey the impression that the figure in the 1917 edition of his *History of Elementary Mathematics* is correct because he used here the term *rectangle,* instead of *square,* to describe one of the 28 feet of which Gerbert spoke. On page 116 of his later second edition of "*A History of Mathematics*" (as well as in a part of the above "reply"), he retained the more definite term "square" for the same purpose, and it seemed only fair to give him credit for concluding finally that the latter term is the better in this connection. This credit seems to have been undue.

If one would like to assume that Gerbert used rectangles of unequal sides as units of measure, one meets with the difficulty of assigning numerical values to these sides as well as to the three concurrent edges of the corresponding rectangular parallelopiped used in measuring solids, to which Gerbert refers in the same letter. It is singular that while the present writer has claimed that Gerbert's letter is of much less scientific value than the references to it in various mathematical histories imply, he does not want to make this letter appear so ridiculous as, it seems to him, the author of the preceding" reply" does, who seems to try to uphold the favorable statements relating to this letter but also to attribute to it by implication a gross want of clearness as regards units of measure.

The foregoing remarks may suffice to make it clear that it is practically certain that Gerbert meant 28 *square feet* when he referred to the diagram to which the figures noted in the first part of the above "reply" relate and hence they seem to justify my claim that most of these figures are inaccurate. It should be noted that by an accurate figure I do not mean that it must be drawn to scale. The sides of the supposed equilateral triangle in the *History of Elementary Mathematics* noted above are not equal, and the squares, or what should be squares, have unequal sides, but I did not find fault with these points. What one seems to have a right to demand of a figure is that if one makes allowances for slight inaccuracies in construction and if one interprets the figure in the light of the subject matter and the text, it should actually illustrate the points in question. In the figure given by Cantor the vertex of the equilateral triangle is about .6 of a unit above the base of the uppermost square while it should be about .06 of a unit above this base. This seemed to me to be too much of a discrepancy to be regarded as correct while the discrepancy in Bubnov's figure seemed to be too slight to be noticed in a small drawing.

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[1] G. A. Miller, *School Science and Mathematics*, vol. 21, 1921, p. 650.

I fail to see why it is assumed in the above "reply" that I missed "completely the essential fact that we do not possess Gerbert's own drawing." It is true that I said that Bubnov gave a correct figure (I did not say that he gave only one figure), but it is to be assumed that the authors of histories in question thought they also gave correct figures, and this assumption is supported by the above "reply." No one, as far as I know, has accused these authors of being ignorant of the fact that their figures are not reproductions of Gerbert's figure. The present writer has never said in public or in private that he thought that we possessed Gerbert's drawing.

The main object that the present writer had in view in making the preceding remarks was to throw some additional light on the nature of the questions involved in the above "reply" to his article. It was there stated that the article "deals admittedly with questions whose satisfactory treatment would require a large amount of space." It can scarcely be supposed or desired that different writers on historical questions should agree as regards all details, since interpretation of language such as the one noted in the last paragraph of the preceding "reply" is often involved, but I hope that enough has now been said to make the main points in question clear to all who may be interested therein.

FINAL NOTE ON THE DISCUSSION BY FLORIAN CAJORI, University of California.

Professor Miller's original contentions are (1) that Bubnov has a "correct" figure, (2) that Cantor's and Günther's are "inaccurate" and "misleading," (3) that Cajori's represents "an isosceles triangle whose base is equal to the altitude," (4) that Gerbert's explanation is "trivial," (5) that Hankel's appreciation of Gerbert's letter is invalid.

Of these only the first two require further comment. My contention that, from Gerbert's text, one cannot tell with certainty what Gerbert's own figure really was, seems borne out not only by the different conjectures made by Cantor, Günther and Bubnov, but also by the fact that Professor Miller, after pronouncing Bubnov's figure "correct," suggested in *School Science and Mathematics*, XXI, p. 652, a different figure of his own. This, however, *does not fit Gerbert's equilateral triangle.* To me, Bubnov's figure is impossible, because it represents to my eye the height as exactly 6. In my judgment, Cantor's figure conveys the truth even to a casual reader by rendering the existence of a fractional excess over 6 strikingly visible. Cantor's and Günther's, though different, appear to me satisfactory illustrations to Gerbert's ingenious letter. In my *History of Mathematics*, 1919, I follow Cantor's explanation; in my *History of Elementary Mathematics*, 1917, I follow Günther's.

Professor Miller says that the main issue is whether the unit of area in Gerbert's letter is the square foot or a rectangle that is not a square. But I have not claimed nor implied that it was other than a square foot. We agree further that "quadratus" sometimes meant "rectangle," for in his own translation of

Gerbert's letter this rendering is given once. His criticism of Günther's figure is valid only if other interpretations of the Latin involved are invalid. Günther's argument, I take it, is simply this, that $\frac{1}{2}a(a+1)$ squares exceed the $\frac{1}{2}a(a+1)$ rectangles drawn, which in turn exceed the area of the given triangle. Finally, I predict that, if Gerbert's own figure is ever brought to light, it will be found to be not Bubnov's nor Miller's, but Cantor's or Günther's.

RECENT PUBLICATIONS.

REVIEWS.

New Mathematical Pastimes. By P. A. MacMahon. Cambridge, University Press, 1921. 8vo. 7 + 116 pages. Price 12 Shillings.

This is an interesting little volume filled with strange and bizarre figures,[1] and punctuated with quaint quotations in verse.[2] The notions are extremely simple (which may surprise readers of some of the author's more serious works) and the language is clear almost to verbiage, as for example (page 50): "The question now before us is the transformation of the set of pieces so that they will no longer be of the same shape and differently coloured or numbered. It proves to be possible to effect this so that the pieces are of different shapes, and are not differently coloured, numbered or otherwise distinguished. In fact, instead of having a set of pieces of the same shape and differently coloured we construct a set of different shapes but of the same colour. The boundary of the assembled pieces now varies in shape with each type and variety but is not otherwise distinguished." The matter is as a whole fresh and pleasingly consecutive. It is far removed in content and treatment from the advanced texts usually issued by the Cambridge University Press.

The author describes his purpose in the following words: "The author of this book has, of recent years, devoted much time and thought to the development of the subject of 'Permutations and Combinations' with which all students are familiar. He has been led, during that time, to construct, for use in the home circle, various sets of pieces, of elementary geometrical shapes based upon these ideas, and he now for the first time brings them together with the object of introducing, in a wider sphere, what he believes to be a pleasant by-path of mathematics which has almost entirely escaped the attention of the well-known writers upon Mathematical Recreations and Amusements. The book differs *in toto* from their works because everything that it contains, with scarcely an exception, is the invention of the author. It is not a bringing together of materials derived from wholly different ideas. From beginning to end it proceeds along one defined

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[1] In its few small pages are one hundred thirty-five separate cuts, averaging some ten distinct designs to a cut.

[2] There are fifty-seven poetic quotations from a wide range of classical sources, scattered through the book. Most of these in their original setting could have had but the most tenuous connection with the thought of this text. They serve merely to entertain the reader by their casual verbal allusions.